PRINCIPLES

OF

ELEMENTARY ALGEBRA.

BY

H. W. KEIGWIN.

BOSTON:
PUBLISHED BY GINN & COMPANY.
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NOTE.

THIS little book is intended as an outline of thorough oral instruction, and is all the "text" I have found necessary to put into my pupils' hands. It should of course be accompanied by a good set of exercises and problems.

Pupils study algebra with much more interest and profit when they are led to discover and to interpret their own formulas and to compose their own rules. I have left much for the teacher and the pupil to do, and have aimed to make the outline brief, accurate, and useful as a text-book.

MATAWAN, NEW JERSEY,
October, 1886.

PRINCIPLES

OF

ELEMENTARY ALGEBRA.

CHAPTER I.

DEFINITIONS, ETC.

1. The numbers of Algebra extend from zero in two opposite directions; those in one direction are called positive, those in the opposite direction negative.

2. Positive and negative numbers are distinguished by prefixing to a positive number the sign $+$, and to a negative number the sign $-$.

The sign $+$ is often omitted when it can be readily understood.

3. The symbols of algebraic numbers are the figures of arithmetic and the letters of the alphabet. Thus, 2, -7, n, $-y$, D, a denote algebraic numbers.

4. The signs $+$, $-$, $\times$, $\div$ have the same general meaning as in arithmetic.

The $\times$ is little used, multiplication being indicated by writing the factors in line. When the factors are the numbers of arithmetic they are sometimes separated by a point. Thus:

$a \times b \times m$ is written abm; 3×5 is written $3 \cdot 5$; $3 \times a + 3 \times 5 \times a + 3 \times 5 \times 7 \times a$ is written $3a + 3 \cdot 5a + 3 \cdot 5 \cdot 7a$.

Division is generally indicated by writing the dividend as the numerator, and the divisor as the denominator of a fraction.

5. The exponential notation is the same in algebra as in arithmetic. Thus: 2^3 means $2 \cdot 2 \cdot 2$; y^4 means $yyyy$. When a factor occurs with no written exponent, the exponent 1 is understood.

6. The radical sign $\sqrt{\ }$ is used with the same meaning in algebra as in arithmetic.

7. The sign of aggregation indicates that the numbers enclosed by the sign are to be taken collectively. The signs used are the parenthesis marks ( ), brackets [ ], braces { }, and vinculum —— or bar |. Thus: $\sqrt{a^2 + b^2}$ means that the sum of a^2 and b^2 is first to be found, and then the square root of that sum is to be found.

$$3\left|\begin{array}{l}+x\\-y\\+z\end{array}\right. \text{ is equal to } 3(x-y+z).$$

8. Any collection of algebraic symbols of number with any of the signs just described is called an *algebraic expression*, or an *expression*, or a *quantity*.

9. The parts of an expression that are connected by + and — signs are called terms.

If a term contain no letter it is called a numerical term.

10. If an expression consist of a single term it is called a monomial; if of two terms, it is called a binomial; if of three terms, a trinomial. Any expression of more than one term is often called a polynomial.

11. A coefficient is a multiplier of any quantity. When several quantities are multiplied together the product of any of them may be considered the coefficient of the remaining product. Thus, in $3ab^2x$, 3 is the coefficient of ab^2x; $3ab^2$ is the coefficient of x; $3ax$ is the coefficient of b^2.

When the term is negative, the coefficient is generally supposed to involve the sign. Thus, in $-5b^2x$ we should generally say that -5 is the coefficient of b^2x.

When a term occurs with no numerical coefficient, the coefficient 1 is understood.

12. Like terms or similar terms are such as have the literal part the same. Thus: abx, $5abx$, $-11abx$ are like.

13. The degree of a term is equal to the number of literal factors it contains, or it is equal to the sum of the exponents of the letters.

14. The numerical value of an algebraic quantity is its value as an arithmetical number. It may be either $+$ or $-$.

15. The absolute value of an algebraic quantity is its value independent of its sign. Thus, in the expression $a^2-8ab+b^2$, if a is 2 and b is 1, the numerical value of the expression is -11, and its absolute value is 11.

16. The degree of an expression is the degree of the highest term in the expression.

17. The sign of equality $=$ is used with the same meaning as in arithmetic.

CHAPTER II.

ADDITION AND SUBTRACTION.

18. Negative numbers are counted in a direction opposite to the direction of positive numbers.

19. Positive numbers are added as in arithmetic. When two or more negative numbers are added, the absolute value of the sum is the same as if they were positive numbers, but the sum is preceded by the sign $-$.

20. When a negative number is added to a positive, the result is found by counting the positive number forward (that is, increasingly), and then counting the negative number backward. The last number counted is called the sum. Thus, to add 4 and -6. Count one, two, three, four; then three, two, one, zero, minus one, minus two; $4-6=-2$. To add m and $-n$; count m units, then count n units in the opposite direction. The last number counted is the sum of m and $-n$, or it is $m+(-n)$.

21. Subtraction may be considered as a counting backward, and if we subtract n units from m units, we count m units; then we count in the opposite direction n units. The last number counted is the difference of m and n, or it is $m-(+n)$.

This operation is the same as the one just described (**20**), and plainly the result $m-(+n)$ is the same as $m+(-n)$. This means that the addition of a negative number is the same as the subtraction of an equal positive number. Both these expressions $m+(-n)$ and $m-(+n)$ are generally written $m-n$.

22. Let n denote any number. If we add a certain number of n's, say a n's, to some other number of n's, say b n's, we may indicate it thus: $an + bn$. It is plain that this sum $an + bn$ must contain n just $a + b$ times; that is,

$$an + bn = (a + b)n.$$

If a and b are arithmetical numbers, an and bn are similar terms. So we get a rule for addition:

Add the coefficients of similar terms, and prefix the sum to the common letter.

Ex. $$\begin{aligned} &2a - 5b + 3c + 7a + 9b - 8c \\ &= (2 + 7)a + (-5 + 9)b + (3 - 8)c \\ &= 9a + 4b - 5c. \end{aligned}$$

23. Any expression or quantity means that the terms are to be combined according to the principles of addition. If there occur a quantity in a parenthesis to be added, it is plain we may remove the sign of aggregation and add the terms at once.

24. Suppose now we have to subtract a binomial. Suppose the expression

$$m - (a + b).$$

If from m we subtract one term at a time, when we have subtracted a (and get $m - a$), we have not taken away enough; we were to subtract a and b, so there remains b to be subtracted. If, then, from m we take a and b one after the other (that is, if we remove the parenthesis and subtract) we shall get $m - a - b$. It follows that

$$m - (a + b) = m - a - b.$$

In the first expression the signs of both a and b are $+$; in the last expression they are both $-$, evidently changed by removing a parenthesis when it is preceded by a minus sign.

Again, suppose the expression

$$m - (a - b).$$

As before, let us subtract a term at a time. When we have taken a from m we have taken too much, for we are required to take $a-b$ (that is, something less than a). Plainly $m-a$ is a result too small by b, and we must add b. Then we have

$$m-a+b.$$

Therefore, $$m-(a-b)=m-a+b,$$

and in this case we have changed the signs of both terms, by removing a parenthesis when preceded by a minus sign.

25. Of course this principle can be extended to any number of terms, and by a similar principle aggregation signs can be introduced into any polynomial expression.

26. From this principle for removing the parenthesis sign, we get the rule for subtraction:

Change the signs of the terms to be subtracted, and proceed as in addition.

27. Often many or all of the signs can be changed mentally, and the rewriting of the whole expression can be avoided.

28. It follows from the last part of **24** that,

$$m-(-b)=m+b.$$

This is true whatever m may be, and if $m=0$ it is still true that,

$$-(-b)=+b.$$

CHAPTER III.

MULTIPLICATION AND DIVISION.

MULTIPLICATION.

29. We have seen (4) that multiplication is indicated by writing the factors successively without the sign $\times$.

Thus: $7ab$ means 7 times a times b; $(a-3b)(x+2y+11)$ means the product of the binomial into the trinomial.

We have seen, too (22), that the product of a monomial into a polynomial is found by multiplying every term of the polynomial by the monomial, and connecting the results.

Thus: $a(bx+ac)=abx+a^2c$.

Evidently if we have several terms in each factor we can get the true product by combining the partial products. We have only to consider the *law of signs.*

30. Let the absolute values of two numbers be denoted by m and n.

(1) If both are positive, the product is positive.

(2) If one, say m, is positive, and the other, n, is negative, we know that if we add any number of $-n$'s, the result will be negative; if we take m such numbers, the product will be $-mn$; that is,

$$(+m)(-n)=-mn.$$

(3) It is assumed in algebra that the order of the factors makes no difference in the value of the product. Therefore, $(+m)(-n)=(-n)(+m)=-mn$. As $(-n)(+m)$ is neg-

ative, $(-m)(+n)$ will also be negative; for the values of m and n are general. Therefore,

$$(-m)(+n) = -mn.$$

(4) If both are negative,

$$(-m)(-n) = -m(-n);$$

writing the coefficient of $-m$, $\quad = -1m(-n);$

from (3) above, first part, $\quad = -1(-n)m;$

dropping the 1, $\quad = -(-n)m;$

by **28**, $\quad = +n(m);$

$$= +mn.$$

Therefore, $\quad (-m)(-n) = +mn.$

31. We infer: If both the signs are plus, or if both are minus, the product is plus. If either one is minus, the product is minus.

32. By arranging the signs in pairs, the sign of the product of any number of factors can be determined, and it will be seen that:

Any number of plus signs gives plus.
An even number of minus signs gives plus.
An odd number of minus signs gives minus.

33. Rule for multiplication: Multiply every term of one polynomial by every term of the other polynomial, and combine the partial products.

DIVISION.

34. In Division we have given a product and one factor to find the other factor.

The process is similar to division in arithmetic, and the law of signs is similar to the law in **31**, as will be seen by considering the cases in **30**.

35. Care should be taken to arrange the terms of both dividend and divisor according to the progressive powers of the same letter. This affords a more satisfactory form in the answer.

CHAPTER IV.

GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE.

GREATEST COMMON DIVISOR.

36. The greatest common divisor of two or more quantities is the quantity of highest degree that will divide every one of them.

The greatest common divisor is denoted by the abbreviation G.C.D. It is also called the greatest common measure (abbreviation, G.C.M.) and the highest common factor (abbreviation, H.C.F.).

37. When the quantities whose G.C.D. is to be found are monomials, it can best be found by inspecting the quantities and determining their factors. The product of the factors common to all is the G.C.D.

38. When the quantities are polynomials, we find the G.C.D. of any two, then of this divisor and a third polynomial, and so on.

39. The method followed is similar to the method in arithmetic:

Divide the polynomial of higher degree by the other polynomial.

Divide the divisor by the last remainder, and so on till there is no remainder.

The last divisor is the G.C.D.

40. To avoid fractional coefficients this rule is slightly modified:

(1) Remove all monomial factors from the polynomials, and save any common factors as a part of the G.C.D.

(2) Remove any factor from any expression in the course of the work when it will facilitate the work.

(3) Whenever the first term of a divisor is not contained an integral number of times in the first term of its dividend, introduce any required factor.

This rule will be proved later.

LEAST COMMON MULTIPLE.

41. The least common multiple of two or more quantities is the quantity of lowest degree that is divisible by every one of them.

The least common multiple, also called the least common measure, is denoted by the abbreviation L.C.M.

42. The L.C.M. of several quantities is plainly the product of the factors occurring in the quantities, each factor taken just times enough so that the multiple will contain every one of the quantities.

43. When the quantities cannot be factored by inspection, the L.C.M. of two quantities is found, then of this multiple and a third quantity, and so on.

44. The L.C.M. of two polynomials is found by multiplying together the G.C.D. of the polynomials and the remaining factor in each polynomial.

The product of the three factors is the L.C.M. of the two polynomials.

CHAPTER V.

FORMULAS AND FRACTIONS.

FORMULAS.

45. A formula is an algebraic expression which, from its frequent application, is of special use. The following are worth memorizing:

(1) $(a+b)^2 = a^2 + 2ab + b^2.$

(2) $a^2 - b^2 = (a+b)(a-b).$

(3) $a^3 + b^3 = (a+b)(a^2 - ab + b^2).$

(4) $a^3 - b^3 = (a-b)(a^2 + ab + b^2).$

(5) $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2),$

$$= (a-b)(a+b)(a^2+b^2),$$

$$= (a-b)(a^3 + a^2b + ab^2 + b^3).$$

(6) $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$

(7) $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$

(8) $a^6 - b^6 = (a^3 - b^3)(a^3 + b^3),$

$$= (a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2);$$

also $= (a^2 - b^2)(a^4 + a^2b^2 + b^4).$

(9) $(x+a)(x+b) = x^2 + (a+b)x + ab.$

FRACTIONS.

46. The laws which govern the treatment of fractions is in general the same as in arithmetic. These cautions are useful:

(1) The sign which precedes the fraction, called the sign of the fraction, may be regarded as belonging to either the numerator or the denominator.

(2) The numerator and the denominator are regarded as "wholes," and are to be treated as if enclosed in parentheses.

This is especially to be considered when a factor precedes or follows the fraction, or when the fraction's sign is minus.

(3) Changing the sign of either the numerator or the denominator changes the sign of the fraction; and

Changing the signs of both numerator and denominator does not change the sign of the fraction.

47. Illustrations:

(1) $$-\frac{a^2-ab+b^2}{m-x}=\frac{-(a^2-ab+b^2)}{m-x}=\frac{a^2-ab+b^2}{-(m-x)}$$
$$=\frac{-a^2+ab-b^2}{m-x}=\frac{a^2-ab+b^2}{x-m}.$$

In this illustration, in all the forms after the first, the *sign of the fraction* (being unwritten) is plus.

(2) $$\frac{a+b}{l}x=\frac{(a+b)x}{l}=\frac{ax+bx}{l}=x\frac{a+b}{l}.$$

CHAPTER VI.

EQUATIONS AND THE SOLUTION OF SIMPLE EQUATIONS.

48. An equation is an algebraic statement that two quantities are equal.

The expression on the left of the sign of equality is called the first member; that on the right is called the second member.

49. An identical equation (or an identity) is an equation which contains only numerical terms; or, it is one which is true whatever values be assigned to the letters. The illustrations and formulas in Chapter V. are instances of identities.

50. An equation of condition is one which is true only for certain values of a letter which represents the *unknown quantity*. These values are called the roots of the equation, and finding them is called solving the equation.

When a root is substituted for the unknown quantity, the equation of condition becomes an identity.

By "equation" is meant an equation of condition, unless otherwise stated or implied.

51. Known quantities are those whose values, it is assumed, are given.

52. Unknown quantities are those whose values are to be found. The final letters of the alphabet are usually reserved for these, though any letters may be used.

53. The degree of an equation is indicated by the largest number of unknown factors which occurs in any term.

Thus: $$3x^2 + ay + 2x^2y = 0$$

is of the third degree, determined by the three unknown factors in the term $2x^2y$. The equation should be free from parentheses, fractions, and radical signs, so far as the unknown quantities are concerned, before its degree is determined.

54. The following axioms (or assumed truths) are useful in transforming equations:

(1) Quantities equal to the same quantity are equal to each other.

(2) If equal quantities be added to (or subtracted from) equal quantities, the results are equal.

(3) If equal quantities be multiplied (or divided) by equal quantities, the results are equal.

(4) If equal powers (or equal roots) of equal quantities be taken, the results are equal.

Axioms (2), (3), and (4) may be summed up in:

Similar operations upon equal quantities give equal results.

55. An equation is not a quantity, and it cannot be multiplied or otherwise treated as a quantity; but we sometimes speak of multiplying an equation, etc., meaning thereby multiplying both members of the equation, etc.

56. Any term may be transposed from one side of an equation to the other by changing its sign. This follows from axiom (2). Thus:

$$a + b - c = x \qquad (1)$$

$$-a \qquad = \quad -a \text{ identity,} \qquad (2)$$

adding member to member, $$b - c = x - a \qquad (3)$$

and a has been transposed from the first to the second member of (1), and its sign is changed.

If we transpose all the terms of (1), we get

$$-x = -a - b + c; \tag{4}$$

therefore, we may change the sign of *every* term in an equation without destroying the equality.

57. Any denominator may be removed from an equation by use of axiom (3); and all the denominators may be removed, or the equation may be "cleared of fractions," by multiplying the equation through by a multiple of the denominators.

58. To solve an equation of the first degree with one unknown quantity (often called a simple equation):

Clear of fractions.

Bring all the terms containing the unknown quantity to the first member, and all the other terms to the second member.

Divide both members by the coefficient of the unknown quantity.

Ex. $$\frac{x}{2} + \frac{a}{b} = mx + l; \tag{1}$$

clearing, $$bx + 2a = 2bmx + 2bl; \tag{2}$$

transposing, $$bx - 2bmx = 2bl - 2a; \tag{3}$$

factoring, $$(b - 2bm)x = 2bl - 2a; \tag{4}$$

dividing, $$x = \frac{2bl - 2a}{b - 2bm} \tag{5}$$

$$= 2\frac{bl - a}{b(1 - 2m)}. \tag{6}$$

CHAPTER VII.

FIRST DEGREE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

59. If we have an equation containing two unknown quantities, it is called indeterminate. It will be seen upon trial that an indefinite number of *pairs of values* for the unknown quantities can be found.

60. If we have two first degree equations containing two unknowns, the equations can in general be solved; that is, such a pair of values for the unknowns can be found that both equations will be satisfied.

61. Elimination is the process of combining two or more equations in such a way as to remove one or more of the unknown quantities. The three common methods are:

I. By addition or subtraction.
II. By comparison.
III. By substitution.

I. By addition or subtraction.

Transform one or both the equations so that the coefficients of one of the unknowns shall be absolutely equal.

Add or subtract the equations according as the equal coefficients have opposite or like signs.

This method is the one most commonly used.

II. By comparison.

Write the equations in the form:

$$x = ay + b, \tag{1}$$

$$x = my + n; \tag{2}$$

then form a new equation by writing the second members equal:

$$ay + b = my + n. \tag{3}$$

This method is often used when the known quantities are literal.

III. By substitution.

Write one of the equations in the form:

$$x = ay + b,$$

and substitute the second member in place of x in the other equation.

This method is often used when one of the equations is of the second degree.

62. It is evident that this process may be extended to three or more first degree equations, containing three or more unknown quantities.

63. From the final equation containing only one unknown quantity, we may get its value (by Ch. VI.), and by successive substitutions the values of the other unknowns may be found.

64. Equations which can be combined for elimination are called simultaneous equations.

65. Equations which can be reduced to the same form are called equivalent equations, or dependent equations.

66. Equations which can be reduced to such form that the coefficients of the unknowns are the same, while the known quantities are different, are inconsistent equations.

67. If we have n equations containing n unknowns, the equations can in general be solved. When solution is impossible, there will occur somewhere in the work either equivalent or inconsistent equations. Neither equivalent nor inconsistent equations can be solved.

CHAPTER VIII.

INVOLUTION AND EVOLUTION.

MONOMIALS.

It is assumed that m and n are positive integers.

68. The nth root of a is a quantity which, raised to the nth power, produces a.

Or, $$(\sqrt[n]{a})^n = a.$$

69. It is evident from multiplication that:

$$a^m a^n = a^{m+n}, \qquad [1]$$

$$(a^m)^n = a^{mn}, \qquad [2]$$

$$(ab)^m = a^m b^m. \qquad [3]$$

These are fundamental equations.

It is also evident that:

$$\sqrt[m]{(a^{mn})} = \sqrt[m]{(a^n)^m} = a^n. \qquad (4)$$

It will now be shown that the mth power of the nth root is equal to the nth root of the mth power; or, that:

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}. \qquad (5)$$

Let $$\sqrt[n]{a} = l. \qquad (6)$$

Then $$(\sqrt[n]{a})^m = l^m; \qquad (7)$$

also, from (6), $$a = l^n, \qquad (8)$$

or, $$a^m = l^{mn}; \qquad (9)$$

therefore, $$\sqrt[n]{a^m} = l^m = (\sqrt[n]{a})^m, \qquad (10)$$

which gives (5) as required.

70. $$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}. \qquad [4]$$

To prove this, from **68** and [3],

$$(\sqrt[n]{a}\sqrt[n]{b})^n=ab;$$

therefore, $$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}.$$

Also, $$\sqrt[n]{\sqrt[m]{a}}=\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}. \qquad [5]$$

To prove this, let $$\sqrt[n]{\sqrt[m]{a}}=l; \qquad (1)$$

then $$a=(l^n)^m=l^{mn}; \qquad (2)$$

therefore, $$l=\sqrt[m]{\sqrt[n]{a}}=\sqrt[n]{\sqrt[m]{a}}=\sqrt[mn]{a}. \qquad [5]$$

Also, $$\sqrt[m]{a}\sqrt[n]{a}=\sqrt[mn]{a^{m+n}}. \qquad [6]$$

To prove this: $$\sqrt[m]{a}=\sqrt[mn]{a^n}, \qquad (7)$$

and $$\sqrt[n]{a}=\sqrt[mn]{a^m}; \qquad (8)$$

therefore, $$\sqrt[m]{a}\sqrt[n]{a}=\sqrt[mn]{a^{m+n}}. \qquad [6]$$

71. A fractional exponent is explained as meaning, by its numerator the power to which the quantity is to be raised, by its denominator the root which is to be taken. Thus:

$$a^{\frac{3}{5}}=\sqrt[5]{a^3}=(\sqrt[5]{a})^3.$$

This explanation will be seen to be consistent with the use of integral exponents.

72. Formulas [6], [5] and [4] mean that [1], [2] and [3] are true when m and n are positive fractions whose numerators are 1; and by a simple extension of **70** it is seen that [1], [2] and [3] are true for all positive values.

73. Negative exponents may be viewed as occurring thus: Divide a^3 successively by increasing powers of a:

$$\frac{a^3}{a}=a^{3-1}=a^2,$$

$$\frac{a^3}{a^2} = a^{3-2} = a^1 = a,$$

$$\frac{a^3}{a^3} = a^{3-3} = a^0, \text{ and as } \frac{a^3}{a^3} = 1$$

we shall explain a^0 as equal to 1.

$$\frac{a^3}{a^4} = a^{3-4} = a^{-1}, \text{ and as } \frac{a^3}{a^4} = \frac{1}{a}$$

we shall explain a^{-1} as equal to $\frac{1}{a}$.

74. The reciprocal of a quantity is 1 divided by that quantity. A negative exponent means the reciprocal of the quantity with an equal positive exponent.

$$a^{-n} = \frac{1}{a^n}. \qquad \frac{1}{a^{-n}} = a^n.$$

It can be shown that [1], [2] and [3] are true when m and n are negative; it follows that they are true universally.

75. A radical quantity is the indicated root of some quantity. When the root cannot be extracted the indicated root is sometimes called a surd.

76. A radical is said to be in its simplest form when every possible operation indicated by the root index (or by the denominator of the fractional exponent) has been performed. A few of the more common reductions are illustrated in these examples:

(1) $\sqrt{a^2 + a^4x^2} = \sqrt{a^2}\sqrt{1 + a^2x^2} = a\sqrt{1 + a^2x^2}.$

(2) $m\sqrt[3]{2 + 7n} = \sqrt[3]{m^3}\sqrt[3]{2 + 7n} = \sqrt[3]{2m^3 + 7m^3n}.$

(3) $\sqrt{\frac{7}{2}} = \frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{7}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{14}}{2} = \frac{1}{2}\sqrt{14}.$

(4) $\sqrt[3]{\frac{7}{2}} = \frac{\sqrt[3]{7}\sqrt[3]{2^2}}{\sqrt[3]{2}\sqrt[3]{2^2}} = \frac{\sqrt[3]{28}}{2} = \frac{1}{2}\sqrt[3]{28}.$

(5) $\sqrt{\dfrac{2+x}{2-x}}=\dfrac{\sqrt{(2+x)(2-x)}}{2-x}=\dfrac{1}{2-x}\sqrt{4-x^2}.$

(6) $\dfrac{\sqrt{2}+\sqrt{x}}{\sqrt{2}-\sqrt{x}}=\dfrac{(\sqrt{2}+\sqrt{x})(\sqrt{2}+\sqrt{x})}{(\sqrt{2}-\sqrt{x})(\sqrt{2}+\sqrt{x})}=\dfrac{2+2\sqrt{2x}+x}{2-x}.$

In (3), (4), (5) and (6) the denominator is cleared of radicals, often a desirable result.

77. Radicals can be combined by addition and subtraction only when the radical parts consist of the same quantity under the same index; such radicals are called similar radicals. Thus:

(1) $\sqrt{8}+\sqrt{32}=\sqrt{4\cdot 2}+\sqrt{16\cdot 2}=2\sqrt{2}+4\sqrt{2}=6\sqrt{2}.$

(2) $\sqrt{8}+\sqrt[3]{8}-\sqrt{2}=2\sqrt{2}+2-\sqrt{2}=2+\sqrt{2};$

$$2+\sqrt{2}=\sqrt{2}(\sqrt{2}+1).$$

78. Radicals can be brought under one sign in multiplication and division only when the factors have the same index. They can always be reduced to the same index. Thus:

$$\sqrt{a}\sqrt[3]{a}=\sqrt[6]{a^3}\sqrt[6]{a^2}=\sqrt[6]{a^3a^2}=\sqrt[6]{a^5}.$$

POLYNOMIALS.

79. If we raise $(a+b)$ to successive powers, we shall find certain laws governing the number of the terms, the exponents of the letters, and the coefficients. Thus:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$
$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$
$$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$$

The law of the number of terms and of the signs and the law of the exponents are obvious; and the symmetrical arrangement of the coefficients is noticeable.

The law of the coefficients is:

If we multiply any coefficient by the exponent of a in that term, and divide the product by a number one greater than the exponent of b in that term, we obtain the next coefficient.

80. $(a+b+c+\cdots\cdots)^2 = a^2+2ab+b^2+2ac+2bc+c^2+\cdots\cdots$

81. By means of the formula,

$$(a+b)^2 = a^2+2ab+b^2,$$

the square root of a polynomial may be found. Thus,

$$a^2+2ab+b^2 = a^2+(2a+b)b.$$

After the term corresponding to a has been found, we find the next term by dividing the "$2ab$ term" by "$2a$." After subtracting "$(a+b)^2$" the "$(a+b)$" is treated as one quantity (say l), and another term in the root is found by the same formula.

82. By means of the formula,

$$\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ &= a^3+(3a^2+3ab+b^2)b,\end{aligned}$$

the cube root of a polynomial may be found. The process is similar to the extraction of square root, and by similar formulas for higher powers the corresponding roots may be obtained.

CHAPTER IX.

RATIO AND PROPORTION.

83. The ratio of two numbers is the quotient of the first divided by the second. The division is indicated by a colon. Thus:

$$a : b = \frac{a}{b}.$$

84. The first quantity (or the first term) (a) is called the antecedent, the second the consequent.

85. An inverse ratio is the reciprocal of the direct ratio. Thus, if $a : b$ is assumed as a ratio it is called the direct ratio of a to b, and $b : a$ is the inverse ratio.

86. A proportion is an equality of ratios.

87. The first and fourth terms are called extremes, the second and third means. Thus: $a : b = m : n$ (also written $a : b :: m : n$) is a proportion; a and n are extremes, b and m means.

88. We have $$a : b :: m : n, \qquad (1)$$

to prove $$a : m :: b : n. \qquad (2)$$

From (1), $$\frac{a}{b} = \frac{m}{n}. \qquad (3)$$

Multiplying by $\frac{b}{m}$, $$\frac{a}{m} = \frac{b}{n}, \qquad (4)$$

or, $$a : m :: b : n. \qquad (2)$$

89. We have $$a : b :: m : n, \tag{1}$$

to prove $$b : a :: n : m. \tag{2}$$

From (1), $$\frac{a}{b} = \frac{m}{n}. \tag{3}$$

Divide 1 by both members,

$$\frac{b}{a} = \frac{n}{m}, \tag{4}$$

or, $$b : a :: n : m. \tag{2}$$

90. We have

$$a : b :: m : n, \tag{1}$$

to prove $$ak + bl : ar + bs :: mk + nl : mr + ns. \tag{2}$$

From (1), $$\frac{a}{b} = \frac{m}{n}. \tag{3}$$

Multiply by $\frac{r}{s}$, add 1 and reduce,

$$\frac{ar + bs}{bs} = \frac{mr + ns}{ns}. \tag{4}$$

Divide 1 by (4) and multiply by $\frac{l}{s}$,

$$\frac{bl}{ar + bs} = \frac{nl}{mr + ns}. \tag{5}$$

Divide 1 by (3),

$$\frac{b}{a} = \frac{n}{m}. \tag{6}$$

Multiply by $\frac{s}{r}$, add 1 and reduce,

$$\frac{bs + ar}{ar} = \frac{ns + mr}{mr}. \tag{7}$$

Divide 1 by (7) and multiply by $\frac{k}{r}$,

$$\frac{ka}{bs+ar}=\frac{km}{ns+mr}. \tag{8}$$

Add (5) and (8),

$$\frac{ak+bl}{ar+bs}=\frac{mk+nl}{mr+ns}, \tag{9}$$

or, $$ak+bl : ar+bs :: mk+nl : mr+ns. \tag{2}$$

By giving special values (*e.g.*, 1 or 0) to one or more of the quantities k, l, r and s, other formulas may be obtained from this.

CHAPTER X.

QUADRATIC EQUATIONS.

WITH ONE UNKNOWN QUANTITY.

91. A quadratic equation is one which contains the second power, and no higher power, of the unknown quantity.

92. A pure quadratic is one which contains the unknown quantity only in the second degree.

93. An affected quadratic is one which contains the unknown in both the first and second degree.

94. A pure quadratic, or a pure equation of any degree, is solved by arranging the unknown quantity on one side of the equation, and the known quantities on the other, and extracting the root of both sides.

A pure quadratic can be reduced to the form

$$x^2 = a, \qquad (1)$$

which gives $$x = +\sqrt{a} \text{ and } -\sqrt{a}; \qquad (2)$$

the two values of x are often abbreviated into $\pm\sqrt{a}$, where the sign $\pm$ means $+$ and $-$. Every root of even index will involve the double sign.

95. If the two members of a pure quadratic equation have opposite signs, the value of the unknown is the indicated square root of a negative quantity. The indicated even root

of a negative quantity is called an *imaginary quantity*, and is so distinguished from the *real quantities* we have so far considered.

If $x^2 = -l$, then $x = \pm\sqrt{-l} = \pm\sqrt{l}\sqrt{-1}$, an imaginary.

96. Every affected (or complete) quadratic may be arranged in the following form, called the general equation of the second degree:

$$x^2 + cx + n = 0. \tag{q}$$

To solve this, transpose the n,

$$x^2 + cx = -n, \tag{1}$$

or,

$$x^2 + 2\frac{c}{2}x = -n. \tag{2}$$

Referring to **45**, (1) it is plain we can make the first member of (2) a square of a binomial by adding $\left(\frac{c}{2}\right)^2$. Adding this we get

$$x^2 + 2\frac{c}{2}x + \left(\frac{c}{2}\right)^2 = -n + \left(\frac{c}{2}\right)^2, \tag{3}$$

$$= \frac{c^2 - 4n}{4}; \tag{4}$$

taking square root,

$$x + \frac{c}{2} = \pm\sqrt{\frac{c^2 - 4n}{4}}, \tag{5}$$

$$x = -\frac{c}{2} \pm \sqrt{\frac{c^2 - 4n}{4}}, \tag{6}$$

$$= \frac{1}{2}[-c \pm \sqrt{c^2 - 4n}]. \tag{7}$$

$$x = \frac{1}{2}[-c + \sqrt{c^2 - 4n}] = \alpha, \tag{a}$$

$$x = \frac{1}{2}[-c - \sqrt{c^2 - 4n}] = \beta. \tag{b}$$

α and β are the roots of the equation.

97. It is plain that

$$\alpha + \beta = -c,$$

and $$\alpha\beta = n;$$

that is, the sum of the roots is equal to minus the coefficient of x in the general equation, and the product of the roots is equal to the term without x.

98. From (a) and (b) we get,

$$x - \alpha = 0, \tag{a_1}$$

$$x - \beta = 0. \tag{b_1}$$

Multiply (a_1) by (b_1), and we get

$$(x - \alpha)(x - \beta) = 0, \tag{1}$$

or, $$x^2 - (\alpha + \beta)x + \alpha\beta = 0, \tag{2}$$

or, $$x^2 + cx + n = 0; \tag{q}$$

that is, $$x^2 + cx + n = (x - \alpha)(x - \beta).$$

Therefore a quadratic expression $x^2 + cx + n$ is factorable into $(x - \alpha)(x - \beta)$, where α and β are the roots of the equation $x^2 + cx + n = 0$.

When α and β are small integers, it is easy to detect these factors by inspection, **45**, (9), and thus to discover the roots at once.

99. Any equation of the form

$$x^{2r} + cx^r + n = 0$$

may be solved in a similar manner.

100. An irrational equation is one in which the unknown quantity is under the radical sign. The solution of an irrational equation often involves some method of reduction illustrated under **76**, or some similar method. Also they often involve the solution of a quadratic.

101. Whether α and β are both real or both imaginary, and if they are both real whether they are both positive or both negative, or one positive and one negative, depends on the relative values of c and n.

WITH TWO UNKNOWN QUANTITIES.

102. There are certain classes of simultaneous equations where one or both the equations are quadratic, that can be solved by elementary methods.

First. When one equation is of the first degree, one of the unknowns may be written in terms of the other (as $x=ay+b$), and substituting this value of x in the quadratic, the resulting equation is a quadratic with one unknown, and can be solved.

Second. When the equations can be reduced to the form

$$\left.\begin{aligned} x^2+a_1x+b_1y+c_1=0 \\ x^2+a_2x+b_2y+c_2=0 \end{aligned}\right\} \tag{1}$$

or to

$$\left.\begin{aligned} xy+a_1x+b_1y+c_1=0 \\ xy+a_2x+b_2y+c_2=0 \end{aligned}\right\} \tag{2}$$

the second degree term can be eliminated, and by substituting as in the first case, we get a quadratic with one unknown, which can be solved.

Third. When the equations are of such form that they may produce an equation of the form

$$x^2+axy+by^2=0, \tag{3}$$

we can obtain a value for x in terms of y by completing the square and reducing; and we can then substitute as above.

Equations which will produce (3) are of the form

$$\begin{cases} x^2 + a_1xy + b_1y^2 + c_1x = 0 & (4) \\ x^2 + a_2xy + b_2y^2 + c_2x = 0, & (5) \end{cases}$$

and

$$\begin{cases} x^2 + a_1xy + b_1y^2 + c_1 = 0 & (6) \\ x^2 + a_2xy + b_2y^2 + c_2 = 0. & (7) \end{cases}$$

By eliminating the first degree or the zero degree term, the form (3) is obtained. Special forms of (4) are,

$$xy + cx = 0, \qquad (8)$$

or,

$$xy + c = 0, \qquad (9)$$

or,

$$x^2 + c = 0. \qquad (10)$$

103. There are special devices which can sometimes be employed, but no general rule for them can be given.

CHAPTER XI.

PROGRESSIONS.

104. A series is a set of terms which succeed each other by some general law.

105. An Arithmetical Progression is a series in which every term is equal to the preceding term, plus or minus some fixed quantity. This fixed quantity is called the common difference. Thus:

$$a,\ a+d,\ a+2d \cdots\cdots,$$

$$-19,\ -16,\ -13 \cdots\cdots,$$

$$3\tfrac{1}{2},\ 3,\ 2\tfrac{1}{2},\ 2 \cdots\cdots,$$

are Arithmetical Progressions.

106. Let $a =$ the first term,

$l =$ the last term,

$n =$ the number of terms,

$d =$ the common difference,

$s =$ the sum of the series;

then from the series $a,\ a+d,\ a+2d \cdots\cdots$ it is evident that the last term equals $a+(n-1)d$;

i.e.,
$$l = a+(n-1)d. \qquad [l]$$

107. $s = a+(a+d)+(a+2d)+\cdots\cdots+(l-d)+l,$ (1)

inverting the order,

$$s = l+(l-d)+(l-2d)+\cdots\cdots+(a+d)+a, \qquad (2)$$

adding,

$$2s = \overline{a+l} + \overline{a+l} + \overline{a+l} + \cdots\cdots + \overline{a+l} + \overline{a+l}. \qquad (3)$$

The number of terms in (1) and (2) is n,

therefore, $$2s = n(a+l), \qquad (4)$$

or, $$s = \frac{n(a+l)}{2}. \qquad [s]$$

108. In $[l]$ and $[s]$ together we have the quantities a, l, n, d, and s; we can eliminate any one, and so express any one in terms of any other three.

109. A Geometrical Progression is a series in which every term bears a fixed ratio to the preceding term. Thus:

$$a,\ ar,\ ar^2,\ ar^3 \cdots\cdots,$$
$$-2,\quad 6,\ -18,\quad 54 \cdots\cdots,$$
$$12,\ 6,\ 3 \cdots\cdots,$$

are Geometrical Progressions.

110. Let r denote the ratio, and use a, l, n, and s as above. Then in the series a, ar, ar^2 ....., the last term equals ar^{n-1},

or, $$l = ar^{n-1}. \qquad [L]$$

111. $$s = a + ar + ar^2 + \cdots\cdots + ar^{n-1}. \qquad (1)$$

Multiply by r,

$$sr = ar + ar^2 + ar^3 + \cdots\cdots + ar^n. \qquad (2)$$

(2) – (1), $$sr - s = ar^n - a, \qquad (3)$$

or, $$s = \frac{a(r^n - 1)}{r-1}. \qquad [S]$$

If r is less than 1, $[S]$ is more conveniently used in the form

$$s = \frac{a(1-r^n)}{1-r}. \qquad [S_1]$$

112. By combining $[L]$ and $[S]$, other formulas may be obtained.

113. If r is a proper fraction, the quantity r^n can be made as small absolutely as we please by increasing the value of n. It can thus be made to approach 0 as nearly as we please, but will always differ slightly from 0. Such a quantity is said to have 0 for its limit. It is plain that if ar^n approaches 0 as a limit, the quantity

$$\frac{a(1-r^n)}{1-r}=\frac{a-ar^n}{1-r}$$

must approach for its limit $\frac{a}{1-r}$. That is, the limit of the sum of the series is $\frac{a}{1-r}$ when n is indefinitely increased. This is sometimes abbreviated in the formula,

$$\lim s_{n \doteq \infty}=\frac{a}{r-1}, \qquad [S']$$

where $\doteq$ means *approaches*, ∞ means *infinity*, and *lim* means *limit of*.

APPENDIX.

A.

PROOF OF RULE FOR FINDING THE G.C.D.

114. The principle of 39 will first be proved. Let M and N denote the given polynomials, q, r and s the quotients, c and d the remainders. Indicate the divisions thus:

$$\begin{array}{l} M)\,N\,(q \\ \quad\underline{Mq} \\ \qquad c)\,M\,(r \\ \qquad\quad \underline{cr} \\ \qquad\qquad d)\,c\,(s \\ \qquad\qquad\quad \underline{ds} \\ \qquad\qquad\qquad 0 \end{array}$$

We will first show that d is *a divisor* of M and N.

$$N = Mq + c, \quad M = cr + d, \quad c = ds;$$

then, $\quad M = dsr + d = d(sr + 1),$

and $\quad N = (dsr + d)q + ds = d(srq + q + s);$

therefore d divides both M and N. We will next show that d is the *greatest* common divisor of M and N.

Every divisor of M and N divides $N - Mq$, that is, c; therefore every divisor of M and N is a divisor of M and c.

Every divisor of M and c divides $M - cr$, that is d; therefore every divisor of M and c is a divisor of c and d.

Therefore every divisor of M and N is a divisor of d. There is no expression higher than d which can divide d; therefore d is the *greatest* common divisor.

115. In regard to **40**, if all the monomial factors are removed from the polynomials before division begins, it is plain that the G.C.D. of the resulting polynomials must contain no monomial factor. Whatever monomials we may introduce or remove during the work can make a difference only in the coefficients, and if at the last we remove all monomial factors from the last divisor, it must be the G.C.D. of the polynomials with which we begin. We are therefore justified in using the suggestions of **40**.

To find the G.C.D. of

$$3x^5 - 10x^3 + 15x + 8 \text{ and } x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6.$$

$$\begin{array}{r} x^5-2x^4-6x^3+4x^2+13x+6)\,3x^5 \qquad -10x^3 \qquad +15x+\ 8(3 \\ \underline{3x^5-6x^4-18x^3+12x^2+39x+18} \\ 6x^4+\ 8x^3-12x^2-24x-10 \end{array}$$

Divide the new divisor by 2 and multiply the new dividend by 3,

$$\begin{array}{r} 3x^4+4x^3-6x^2-12x-5)\,3x^5-\ 6x^4-18x^3+12x^2+39x+18(x \\ \underline{3x^5+\ 4x^4-\ 6x^3-12x^2-\ 5x \qquad} \\ -10x^4-12x^3+24x^2-44x+18 \end{array}$$

Divide this remainder by 2, multiply it by 3, and use the same divisor again,

$$\begin{array}{r} -15x^4-18x^3+36x^2+66x+27(-5 \\ \underline{-15x^4-20x^3+30x^2+60x+25} \qquad \\ 2x^3+\ 6x^2+\ 6x+\ 2 \qquad \end{array}$$

Divide the new divisor by 2,

$$\begin{array}{r} x^3+3x^2+3x+1)\,3x^4+4x^3-\ 6x^2-12x-5(3x-5 \\ \underline{3x^4+9x^3+\ 9x^2+\ 3x \qquad} \qquad\quad \\ -5x^3-15x^2-15x-5 \qquad\quad \\ \underline{-5x^3-15x^2-15x-5} \qquad\quad \end{array}$$

x^3+3x^2+3x+1 is the G.C.D. required.

B.

ILLUSTRATIONS OF SQUARE ROOT AND CUBE ROOT.

116. To find the square root of $4x^4 - 12x^3 + 5x^2 + 6x + 1$.

$$
\begin{array}{lrl}
 & 4x^4 - 12x^3 + 5x^2 + 6x + 1 & (2x^2 - 3x - 1 \\
 & 4x^4 & \\
\cline{2-2}
4x^2 - 3x) & -12x^3 + 5x^2 \phantom{{}+6x+1} & \\
 & -12x^3 + 9x^2 \phantom{{}+6x+1} & \\
\cline{2-2}
4x^2 - 6x - 1) & -4x^2 + 6x + 1 & \\
 & -4x^2 + 6x + 1 & \\
\cline{2-2}
\end{array}
$$

After the second subtraction we have subtracted $(2x^2 - 3x)^2$. If we call $(2x^2 - 3x) = l$, the second trial divisor is $2l$, the second complete divisor is $(2l - 1)$; in the third subtraction we subtract $-1 \cdot (2l - 1)$, and we may regard the polynomial as represented by $l^2 - 2l + 1 = l^2 - 1 \cdot (2l - 1)$; which accords with the formula in **81**.

117. $\sqrt[3]{x^6 - 6x^5 + 40x^3 - 96x - 64} = ?$

$$
\begin{array}{lrl}
 & x^6 - 6x^5 \qquad\qquad + 40x^3 - 96x - 64 & (x^2 - 2x - 4 \\
 & x^6 & \\
\cline{2-2}
3x^4 - 6x^3 + 4x^2) & -6x^5 \qquad\qquad + 40x^3 & \\
 & -6x^5 + 12x^4 - 8x^3 & \\
\cline{2-2}
3x^4 - 12x^3 + 12x^2 & -12x^4 + 48x^3 - 96x - 64 & \\
\quad -12x^2 + 24x + 16 & -12x^4 + 48x^3 - 96x - 64 & \\
\cline{1-2}
3x^4 - 12x^3 \qquad + 24x + 16 & & \\
\end{array}
$$

We may here regard $(x^2 - 2x) = l$, and it will be seen that this process accords with the formula for $(a + b)^3$ in **82**.

C.

ILLUSTRATIONS OF EXAMPLES SOLVED LIKE QUADRATICS.

118. $x+4\sqrt{x}=21.$ Let $\sqrt{x}=y$, then $x=y^2$,

$$y^2+4y=21. \tag{1}$$

$$y(=\sqrt{x})=3 \text{ and } =-7. \tag{2}$$

$$x=9 \text{ and } =49. \tag{3}$$

119. $x+\sqrt{5x+10}=8.$

$$\sqrt{5x+10}=8-x, \tag{1}$$

$$5x+10=(8-x)^2=64-16x+x^2, \tag{2}$$

$$x=18 \text{ and } =3. \tag{3}$$

If we substitute these values of x in the given equation, we shall find that 18 is not a root of the equation. 18 is a root of the equation $x-\sqrt{5x+10}=8$, and we should remember that the sign $\sqrt{\ }$ strictly signifies $\pm\sqrt{\ }$. If we drop a part of the symbol $\pm$, and solve the equation by squaring, we must test the answers to be sure that we have true roots.

120. $$x^2-7x+\sqrt{x^2-7x+18}=24, \tag{1}$$

add 18 to (1), $$x^2-7x+18+\sqrt{x^2-7x+18}=42. \tag{2}$$

Call $y=\sqrt{x^2-7x+18}$,

$$y^2+y=42, \tag{3}$$

$$y=6 \text{ and } =-7. \tag{4}$$

Ans. $x=9$; $x=-2$.

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